MATB41 Week 8 Notes

1. Quadratic Form:
   - A homogeneous polynominal of degree two.

     I.e. Every non-zero terms are all of degree two.

     E.g. $x^2 + xy + y^2$

   - All quadratic forms can be represented by matrix multiplican in the form of $\vec{x}^T A \vec{x}$.

   - E.g. $f(x,y) = x^2 + 2xy + y^2$

     $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\vec{x}^T = [x \ y]$ $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

     $f(x,y) = \vec{x}^T A \vec{x}$

     $= [x \ y] \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

     Here, A is the upper triangular coefficient matrix.

     We can also express $f(x,y)$ as $x^2 + (xy) + (yx) + y^2$.

     $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\vec{x}^T = [x \ y]$ $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

In the first form, $A = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & & & \vdots \\ 0 & & & U_{nn} \end{bmatrix}$

where each U_{ij} is a coefficient.

In the second form, $A = \begin{bmatrix} U_{11} & \frac{1}{2}U_{12} & \cdots & \frac{1}{2}U_{1n} \\ \frac{1}{2}U_{12} & U_{22} & \cdots & \frac{1}{2}U_{2n} \\ \vdots & & & \\ \frac{1}{2}U_{n1} & \cdots & \cdots & U_{nn} \end{bmatrix}$

Here, A is a symmetric matrix, and is called the symmetric coefficient matrix.

- Since we know that all symmetric matrixes are diagonalizable and we know that all quadratic forms can be written as $\vec{x}^T A \vec{x}$, we can diagonalize every quadratic form.

  Given a quad form $f(x) = \vec{x}^T A \vec{x}$, we let C be an orthogonal diagonalizing matrix. Then, the substitution $\vec{x} = C\vec{t}$ is called a diagonalizing sub.

- To diagonalize a quad form:
  1. Find the symmetric coefficient matrix A.
  2. Find the eigenvalues of A.
  3. Find an orthonormal basis consisting of eigenvectors of A.
  4. Use this orthonormal basis to form C and sub $\vec{x} = C\vec{t}$ giving the diagonalizing substitution.
  5. In diagonal form, the quad form becomes $\sum_{i=1}^{n} \lambda_i (t_i)^2 = \vec{t}^T D\vec{t} = \lambda_1 t_1^2 + \dots + \lambda_n t_n^2$

E.g. Diagonalize $x^2 + 2xy + 6xz + 3y^2 + 2yz + z^2$

Soln:

1. $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

2. $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda)(-2-\lambda)$$

$\lambda_1 = 5 \rightarrow$ Eigenvector $\vec{v_1} = [1,1,1]$

$\lambda_2 = 2 \rightarrow$ Eigenvector $\vec{v_2} = [1,-2,1]$

$\lambda_3 = -2 \rightarrow$ Eigenvector $\vec{v_3} = [-1,0,1]$

3. Normalizing the eigenvectors, we get

$$\vec{q_1} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \vec{q_2} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix}, \quad \vec{q_3} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

4. $$C = \begin{bmatrix} | & | & | \\ \vec{q_1} & \vec{q_2} & \vec{q_3} \\ | & | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = C^T A C = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

5. Subbing in $\vec{x} = C\vec{t}$, where $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$, we get:

$$\begin{aligned} f(x,y,z) &= \vec{x}^T A \vec{x} \\ &= (C\vec{t})^T A (C\vec{t}) \\ &= \vec{t}^T C^T A C \vec{t} \\ &= \vec{t}^T D \vec{t} \\ &= 5t_1^2 + 2t_2^2 - 2t_3^2 \end{aligned}$$

- Principal Axis Thm:
  If A is a symmetric $n \times n$ matrix, then there exists an orthogonal matrix C that transforms the quad form $\vec{x}^T A \vec{x}$ into $\vec{t}^T D \vec{t}$ with no product terms, where $\vec{x} = C\vec{t}$.

- A quad form $f(\vec{x}) = \vec{x}^T A \vec{x}$ is said to be
  1. Postive Definite if $f(\vec{x}) > 0 \ \forall \vec{x} \neq \vec{0}$
  2. Negative Def if $f(\vec{x}) < 0 \ \forall \vec{x} \neq \vec{0}$
  3. Indefinite if $f(\vec{x})$ has both pos and neg values.

- Thm: If A is a symmetric matrix, then:
  1. $\vec{x}^T A \vec{x}$ is pos def iff all eigenvalues of A are positive.
  2. $\vec{x}^T A \vec{x}$ is neg def iff all eigenvalues of A are negative.
  3. $\vec{x}^T A \vec{x}$ is indef iff A has at least one positive eigenvalue and at least one negative eigenvalue.

- Thm: The quad form given by a symm matrix A is pos def iff the det of A_k, every $k \times k$ submatrix, is positive.

E.g.
$$A = \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix} \end{array}$$

$$\left.\begin{aligned} \det(A_1) &= |2| = 2 > 0 \\ \det(A_2) &= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0 \\ \det(A_3) &= 1 > 0 \end{aligned}\right\} \therefore \vec{x}^T A \vec{x} \text{ is pos def}$$

- Corollary to the last Thm:
  The quad form given by a symm $n \times n$ matrix is neg def iff the sign of $\det(A_k)$ is given by $(-1)^k$.

2. Taylor Series:

E.g. Find an approx value for the expression $\cos(0.2)\ln(1.01)$ using the 4^{th}-order Taylor Series.

Soln:

$T_4(x,y) = x^2 - x^4 - x^3y - \frac{1}{2}x^2y^2$, $|x|<1$, $y \in \mathbb{R}$

$f(x,y) \approx T_4(x,y)$

Since $1+x^2 = 1.01$, $x = 0.1$

$y = 0.2 - x$

$= 0.1$

$$\begin{aligned}\cos(0.2)\ln(1.01) &\approx f(0.1, 0.1)\\ &\approx T(0.1, 0.1)\\ &= (0.1)^2 - (0.1)^4 - (0.1)^4 - \tfrac{1}{2}(0.1)^4\\ &= 0.00975\end{aligned}$$

3. Mean Value Theory (MVT):

- Let f be diff on its domain, including on the line joining $\vec{a}$ to $\vec{b}$. Then, $f(\vec{b}) - f(\vec{a}) = Df(\vec{z}) \cdot (\vec{b} - \vec{a})$ for some $\vec{z}$ on the line joining $\vec{a}$ to $\vec{b}$.

  If $Df(\vec{x}) = \vec{0}$, $\forall \vec{x}$, then, f is constant.

- E.g. Let $f(x,y) = x^2 + y^2$.
  Verify MVT for $\vec{a} = [1,1]$, $\vec{b} = [2,3]$.

Soln:

Let $\vec{z} = f(x,y)$

$Df(\vec{z}) = \nabla f = [2x, 2y]$

$f(\vec{b}) - f(\vec{a}) = f(2,3) - f(1,1) = 2^2 + 3^2 - 1^2 - 1^2 = 11$

$Df(\vec{z}) \cdot (\vec{b} - \vec{a}) = [2x, 2y] \cdot [1,2] = 2x + 4y$

$2x + 4y = 11$ ①

$\frac{y-1}{x-1} = \frac{3-1}{2-1} = 2$

$y = 2x - 1$ ②

Solving for x and y, we get
$x = \frac{3}{2}$, $y = 2$.

$\therefore f(2,3) - f(1,1) = Df(\frac{3}{2}, 2) \cdot (1, 2)$

4. Max/ Min Values:

- Def: A local/relative min point of $f: U \subset R^n \to R$ is a point $x_0 \in U$ s.t. $f(x_0) \leq f(x)$ $\forall x$ in the neighbourhood of x_0. $f(x_0)$ is the corresponding local/relative min value.

- Def: A local/ relative max point of $f: U \subset R^n \to R$ is a point $x_0 \in U$ s.t. $f(x_0) \geq f(x)$ $\forall x$ in the neighbourhood of x_0. $f(x_0)$ is the corresponding local/relative max value.

- If x_0 is a local min or local max, then it is a local extrenum and $f(x_0)$ is a local extrenum value.

5. Critical Point:

- Def: A point, y_0, is a critical point if either f is NOT diff at Y_0 or $Df(Y_0) = 0$.

6. First Derivative Test:

- Def: If $U \subset \mathbb{R}^n$ is open, $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is diff and x_0 is a local ext, then x_0 is a crit point.

  I.e. All partial derivatives of f vanish at x_0.

$$\frac{\partial f}{\partial x_1}(x_0) = 0, \ldots, \frac{\partial f}{\partial x_n}(x_0) = 0$$

- E.g. Find the critical points of $f(x,y) = xy(x-2)(y+3)$

Soln:

$$\frac{\partial f}{\partial x} = [y+3][y(x-2) + xy]$$
$$= [y+3][2y][x-1]$$

$$\frac{\partial f}{\partial y} = (x)(x-2)(2y+3)$$

To find the crit points,

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$(2y)(x-1)(y+3) = 0 \to y=0,\ x=1,\ y=-3$$
$$(x)(x-2)(2y+3) = 0$$

1. At $y=0 \Rightarrow (x)(x-2)(2(0)+3)=0$

$x=0$ or $x=2$

$(0,0)$ and $(2,0)$ are crit points.

2. At $x=1 \Rightarrow (1)(1-2)(2y+3)=0$

$y=-3/2$

$(1,-3/2)$ is a crit point.

3. At $y=-3 \Rightarrow (x)(x-2)(2(-3)+3)=0$

$-3(x)(x-2)=0$

$x=0$ or $x=2$

$(2,-3)$ and $(0,-3)$ are crit points.

In total, we have 5 crit points: $(0,0)$, $(0,-3)$, $(1,-3/2)$, $(2,0)$ and $(2,-3)$.

Note: We set $\frac{\partial f}{\partial y}=0$ and found the crit points by plugging the zeroes of $\frac{\partial f}{\partial y}$ into $\partial f/\partial x$. We can get the same crit points by doing the same thing but using the other partial derivative.

$(x)(x-2)(2y+3)=0 \rightarrow x=0,\ x=2,\ y=-\frac{3}{2}$

1. At $x=0 \rightarrow (2y)(0-1)(y+3)=0$

   $y=0$ or $y=-3$

   $(0,0)$ and $(0,-3)$ are crit points.

2. At $x=2 \rightarrow (2y)(2-1)(y+3)=0$

   $y=0$ or $y=-3$

   $(2,0)$ and $(2,-3)$ are crit points

3. At $y=-\frac{3}{2} \rightarrow (-3)(x-1)(\frac{3}{2})=0$

   $x=1$

   $(1,-\frac{3}{2})$ is a crit point.

$(0,0)$, $(0,-3)$, $(1,-3/2)$, $(2,0)$, $(2,-3)$ are the crit points. They are the same crit points we got before.

- Def:

  A crit point that is NOT a local ext is a saddle point.

- E.g. Find all the crit points of $f(x,y)=x^2-y^2$ and see if they are saddle points.

$\frac{\partial f}{\partial x}=2x \quad \frac{\partial f}{\partial y}=2y$ Setting both to 0, we get $(0,0)$ as our only crit point.

$f(0,0) = 0$

$f(x,0) = x^2 \geq 0$

$= f(0,0)$

$f(0,y) = -y^2 \leq 0$

$= f(0,0)$

$\therefore$ $(0,0)$ is a saddle point.

7. Hessian Matrix:

- Def: If $f: U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is of class C^3 and the Hessian of f at x_0 is the quad function of h given by

$$Hf(x_0)(\vec{h}) = \frac{1}{2!} \sum_{i_1, i_2 = 1}^{n} h_{i_1} h_{i_2} \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x_0)$$

Note: $\vec{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$

$$= \frac{1}{2} \vec{h}^T \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x_0) & \cdots & \\ \vdots & & \\ & & \frac{\partial^2 f}{\partial x_n \partial x_n}(x_0) \end{bmatrix} \vec{h}$$

Hessian Matrix

I.e. The Hessian Matrix is a square matrix of the 2^{nd} partial derivatives

Note: By equality of mixed partials, the Hessian Matrix is symmetric.

$$Hf(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Note: $Hf(x_0)(\vec{h})$ equals to the 3^{rd} term in the Taylor Expansion of f about x_0.

$$f(x_0+h) = f(x_0) + \sum_{i_1=1}^{n} h_{i_1} \frac{\partial f}{\partial x_{i_1}}(x_0) + \underbrace{\frac{1}{2!} \sum_{i_1, i_2=1}^{n} h_{i_1} h_{i_2} \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x_0)}+ \dots$$

8. Second-Derivative Test:
   - Def: Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^3, $x_0 \in$ an open disk $\subset U$ be a crit point of f.

     If $Hf(x_0)$ is:
     1. Pos Def, then $(x_0, f(x_0))$ is a local min of f.
     2. Neg Def, then $(x_0, f(x_0))$ is a local max of f.
     3. Neither, then $(x_0, f(x_0))$ is a saddle point.

Note: If $\det(Hf(x_0)) = 0$, then it is of degenerate type.

- Second-Derivative Test for Functions of 2 Vars:
  Let $f(x,y)$ be of class C^2, $(x_0, y_0) \in$ an open disk $\subset U$ be a crit point of f satisfying $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.
  Let $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2$

  If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local min value at (x_0, y_0).

  If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local max value at (x_0, y_0).

  If $D(x_0, y_0) < 0$, then f has a saddle point at (x_0, y_0).

  If $D(x_0, y_0) = 0$, then the test is inconclusive.

E.g. Let $f(x,y,z) = x^2+y^2+z^2-2xy$. Find all the crit points of f and determine if they are local min, local max, saddle points or none.

Soln:

$$\frac{\partial f}{\partial x} = 2x-2yz = 0 \rightarrow x = yz$$

$$\frac{\partial f}{\partial y} = 2y-2xz = 0 \rightarrow y = xz$$

$$\frac{\partial f}{\partial z} = 2z-2xy = 0 \rightarrow z = xy$$

If $x=0$, $y=0$, or $z=0$, we get $x=y=z=0$. Therefore, $(0,0,0)$ is a crit point.

If $x \neq 0$, then $y \neq 0$ and $z \neq 0$.

$$xyz = (xyz)^2$$
$$xyz = 1$$
$$x^2 = 1 \rightarrow x = \pm 1$$
$$y^2 = 1 \rightarrow y = \pm 1$$
$$z^2 = 1 \rightarrow z = \pm 1$$

$\therefore$ The crit points are $(1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$, $(1,-1,-1)$ and $(0,0,0)$.

$$Hf(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2z & -2y \\ -2z & 2 & -2x \\ -2y & -2x & 2 \end{bmatrix}$$

$$Hf(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Since $\lambda_1 = \lambda_2 = \lambda_3 = 2 > 0$, it is pos def.

$\therefore$ (0,0,0) is a local min.

$$Hf(1,1,1) = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

$\lambda_1 = -2$

$\lambda_2 = \lambda_3 = 4$

$\therefore$ (1,1,1) is a saddle point

$Hf(-1,-1,1)$ and the rest have eigenvalues of $-2, 4, 4$. $\therefore$ They are all saddle points.

E.g. $f(x,y) = xy(x-2)(y+3)$ has 5 crit points. They are $(0,0)$, $(2,0)$, $(1,-\frac{3}{2})$, $(0,-3)$ and $(2,-3)$. Use the Second-Derivative Test to classify them.

Soln:

$f_{xx} = (2y)(y+3)$

$f_{yy} = (2x)(x-2)$

$f_{xy} = f_{yx} = 2(2y+3)$

$D(x_0, y_0) = 4(x_0)(y_0)(x_0-2)(y_0+3) - 4(2y_0+3)^2$

$D(0,0) = -36$ (Saddle Point)

$D(2,0) = -36$ (Saddle Point)

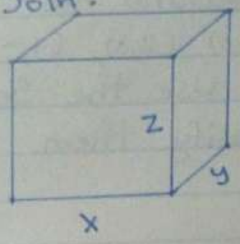
$D(0,-3) = -36$ (Saddle Point)

$D(2,-3) = -36$ (Saddle Point)

$D(1,-\frac{3}{2}) = 9$, $f_{xx}(1,-\frac{3}{2}) < 0 \therefore$ (Local max)

E.g. A shipping company handles rectangular boxes provided that the sum of the length, width and height does not exceed 96 in. Find the dimensions of the box that meets this condition and has the largest volume.

Soln:



$x+y+z \le 96$

$V = xyz$

$$\left.\begin{array}{l} x, y, z \ge 0 \\ x, y, z \le 96 \end{array}\right\} \text{Boundary}$$

Let $z = 96-x-y$

$V(x,y) = (x)(y)(96-x-y)$

$$\frac{\partial f}{\partial x} = 96y - 2xy - y^2 = 0 \longrightarrow y(96-2x-y)=0$$

$y=0$ or

$96-2x-y=0$

$$\frac{\partial f}{\partial y} = 96x - x^2 - 2xy = x(96-x-2y)$$

1. When $y=0$, we get $(0,0)$ and $(96,0)$.

2. When $96-2x-y=0$, $y=96-2x$.

$x(96-x-2(96-2x))=0$

$x(96-x-192+4x)=0$

$x(x-32)=0$

$x=0 \longrightarrow y=96 \quad (0,96)$

$x=32 \longrightarrow y=32 \quad (32,32)$

The crit points are $(0,0)$, $(0,96)$, $(96,0)$, and $(32,32)$.

From the crit points, only (32, 32) will not get a volume of 0, so we choose that.

$V_{xx} = -2y$, $V_{yy} = 2x$, $V_{xy} = V_{yx} = 96 - 2x - 2y$

$$D(x, y) = V_{xx} \cdot V_{yy} - (V_{xy})^2$$
$$= 3072 > 0$$

$$V_{xx}\big|_{(32,32)} = -64 < 0$$

$\therefore$ The vol reaches its max value at
$x = 32$, $y = 32$, $z = 32$

Note: The max vol or max area usually occurs when the dimensions are closest to each other.

9. Open Sets:
- Recall: Let x_0 be a point in R^n.
An open disk or open ball, denoted as $D_r(x_0)$, is $\{x \in R^n \mid \|x - x_0\| < r\} \subset R^n$

In R^1, an open set is an open interval.
In R^2, it's an open disk.
In R^3, it's an open ball.

10. Interior and Boundary Points:

- Def: Let $U \subset \mathbb{R}^n$. A point $x_0 \in U$ is an interior point of U if $D_r(x_0) \subset U$ for some r.

- Def: Points that are not interior points are boundary points.

- The set of boundary points is denoted ∂U. If every point in U is an interior point, U is said to be open.

- U is closed if $\mathbb{R}^n - U$ is open.

- $U \subset \mathbb{R}^n$ is bounded if U can be contained in an open ball, $D_r(0)$, for a sufficiently large R or if $\|x\| < M$, for some $M \in \mathbb{R}$, $\forall x \in U$.

- A closed and bounded set in $\mathbb{R}^n$ is compact.

- Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is defined on a set U in $\mathbb{R}^n$. A point $x_0 \in U$ is a global (absolute) min of f on U if $f(x_0) \leq f(x)$ $\forall x \in U$. x_0 is a global max of f on U if $f(x_0) \geq f(x)$ $\forall x \in U$. If x_0 is either of these, it's a global ext.